

Towards a relativistic statistical theory

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In special relativity the mathematical expressions, defining physical observables as the momentum, the energy etc, emerge as one parameter (light speed) continuous deformations of the corresponding ones of the classical physics. Here, we show that the special relativity imposes a proper one parameter continuous deformation also to the expression of the classical Boltzmann-Gibbs-Shannon entropy. The obtained relativistic entropy permits to construct a coherent and selfconsistent relativistic statistical theory [Phys. Rev. E **66**, 056125 (2002); Phys. Rev. E **72**, 036108 (2005)], preserving the main features (maximum entropy principle, thermodynamic stability, Lesche stability, continuity, symmetry, expansivity, decisivity, etc.) of the classical statistical theory, which is recovered in the classical limit. The predicted distribution function is a one-parameter continuous deformation of the classical Maxwell-Boltzmann distribution and has a simple analytic form, showing power law tails in accordance with the experimental evidence.

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I. INTRODUCTION

In the ordinary relativistic statistical mechanics the entropy is assumed to have the same form of the entropy of classical statistical mechanics. Then, starting from the Boltzmann-Gibbs-Shannon entropy, the maximum entropy principle yields an exponential distribution which exactly reproduces the Maxwell-Boltzmann distribution of classical statistical mechanics in the rest frame. Unfortunately the Maxwell-Boltzmann distribution fails to explain the spectrum of cosmic rays, which undoubtedly represent the most famous and important relativistic particle system [1, 2]. This spectrum has a very large extension (13 decades in energy and 33 decades in particle flux), exhibiting a power law asymptotic behavior. Similar power law asymptotic behaviors of the distribution function have been observed also in other relativistic systems including plasmas [3] and multiparticle production processes [4]. Hence for relativistic systems, the experimental evidence suggests a non-exponential distribution function with power law tails. In the last 40 years, empirical non-exponential distributions with power law tails have been systematically used in high energy plasmas.

A non-exponential distribution can be originated exclusively by an entropy manifestly different from the Boltzmann-Gibbs-Shannon one. In order to propose theories based on non-exponential statistical distributions, one can follow different paths, that allow to construct, other coherent and selfconsistent statistical theories besides the classical statistical mechanics [5, 6, 7, 8, 9, 10, 11, 12].

We recall that any relativistic formula, defining a physical quantity, e.g. momentum, energy, etc, can be viewed as a one parameter (light speed) generalization or deformation of the corresponding classical formula.

Consequently, it is natural to ask if the entropy of a relativistic system could be a one parameter generalization of the Boltzmann-Gibbs-Shannon entropy as well.

The main goal of the present contribution is to show that the one-particle relativistic dynamics imposes a one parameter generalization of the Boltzmann-Gibbs-Shannon entropy. The new entropy permits the construction of a statistical mechanics which efficiently describes the relativistic many-particle systems.

II. COMPOSITION LAWS IN SPECIAL RELATIVITY

We introduce the dimensionless variables momentum q , velocity u and total energy \mathcal{E} in place of the corresponding physical variables p , v and E through

$$\frac{v}{u} = \frac{p}{mq} = \sqrt{\frac{E}{m\mathcal{E}}} = \kappa c = v_* \quad , \quad (1)$$

being c the light speed and κ a dimensionless parameter. Furthermore we impose for the velocity v_* that $\lim_{c \rightarrow \infty, \kappa \rightarrow 0} v_* = v_*^\infty$ with $v_*^\infty = \text{finite}$, in order to preserve the validity of the dimensionless variable definitions, also in the classical limit. Within the special relativity it is immediate to express the deformation effects, due to the finite value of the light speed, in terms of the dimensionless parameter κ :

$$q(u) = \frac{u}{\sqrt{1 - \kappa^2 u^2}}, \quad (2)$$

$$u(q) = \frac{q}{\sqrt{1 + \kappa^2 q^2}}, \quad (3)$$

$$\mathcal{E}(q) = \frac{1}{\kappa^2} \sqrt{1 + \kappa^2 q^2}, \quad (4)$$

$$q(\mathcal{E}) = \sqrt{\kappa^2 \mathcal{E}^2 - \frac{1}{\kappa^2}}, \quad (5)$$

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$$\mathcal{E}(u) = \frac{1}{\kappa^2 \sqrt{1 - \kappa^2 u^2}}, \quad (6)$$

$$u(\mathcal{E}) = \frac{\sqrt{\kappa^2 \mathcal{E}^2 - 1/\kappa^2}}{\kappa^2 \mathcal{E}}. \quad (7)$$

Consider in the one-dimensional frame \mathcal{S} two identical particles of rest mass m . We suppose that the first particle moves towards right with velocity u_1 while the second particle moves towards left with velocity u_2 . The momenta of the two particles are given by $q_1 = q(u_1)$ and $q_2 = q(u_2)$ respectively, where $q(u)$ is defined through Eq. (2). The energies of the two particles are given by $\mathcal{E}_1 = \mathcal{E}(u_1)$ and $\mathcal{E}_2 = \mathcal{E}(u_2)$ respectively, with $\mathcal{E}(u)$ defined by Eq. (6). We consider now the same particles in a new frame \mathcal{S}' which moves at constant speed u_2 towards left with respect to the frame \mathcal{S} . In this new frame, which is the rest frame of the second particle, we have that the two particles have velocities given by $u'_1 = u_1 \oplus u_2$ and $u'_2 = 0$ respectively, being

$$u_1 \oplus u_2 = \frac{u_1 + u_2}{1 + \kappa^2 u_1 u_2}, \quad (8)$$

the well known relativistic additivity law for the dimensionless velocities. In the same frame \mathcal{S}' the particle momenta are given by $q'_1 = q(u'_1)$ and $q'_2 = 0$, respectively. Analogously in \mathcal{S}' the particle energies of the two particle in \mathcal{S}' results $\mathcal{E}'_1 = \mathcal{E}(u'_1)$ and $\mathcal{E}'_2 = 1/\kappa^2$. A very interesting result follows from the relation $\mathcal{E}'_2 = 1/\kappa^2$ regarding the physical meaning of the deformation parameter κ . It is evident that the quantity $1/\kappa^2$ represents the dimensionless rest energy of the relativistic particle.

Let us pose now the following questions: is it possible to obtain the value of the momentum q'_1 (of the energy \mathcal{E}'_1) starting directly from the values of the momenta q_1 and q_2 (of the energies \mathcal{E}_1 and \mathcal{E}_2) in the frame \mathcal{S} . The answers to the above questions are affirmative.

First we consider the case of the relativistic momentum q'_1 and write it as $q'_1 = q(u'_1) = q(u_1 \oplus u_2)$. In ref. [11] it has been shown that $q'_1 = q_1 \oplus q_2$ being

$$q_1 \oplus q_2 = q_1 \sqrt{1 + \kappa^2 q_2^2} + q_2 \sqrt{1 + \kappa^2 q_1^2}, \quad (9)$$

the κ -sum of relativistic momenta. In words, the relativistic momentum q'_1 of the first particle, in the rest frame of the second particle \mathcal{S}' , is the κ -deformed sum of the momenta q_1 and q_2 of the two particles in the frame \mathcal{S} . The κ -sum of the relativistic momenta and the relativistic sum of the velocities are intimately related, and reduce both to the standard sum as the velocity c approaches to infinity (or equivalently the parameter κ approaches to zero). The deformations in the above sums, in both cases, are relativistic effects and are originated from the fact that c has a finite value.

We consider now the total energy \mathcal{E}'_1 of the first particle in the frame \mathcal{S}' . Clearly it results $\mathcal{E}'_1 = \mathcal{E}(u'_1) = \mathcal{E}(u_1 \oplus u_2)$. In ref. [12] it is shown that one can calculate \mathcal{E}'_1 starting directly from the values, in the frame

\mathcal{S} , of the total energies \mathcal{E}_1 and \mathcal{E}_2 . It is obtained $\mathcal{E}'_1 = \mathcal{E}_1 \oplus \mathcal{E}_2$ where the composition law of the dimensionless relativistic total energies is defined through

$$\mathcal{E}_1 \oplus \mathcal{E}_2 = \kappa^2 \mathcal{E}_1 \mathcal{E}_2 + \frac{1}{\kappa^2} \sqrt{(\kappa^4 \mathcal{E}_1^2 - 1)(\kappa^4 \mathcal{E}_2^2 - 1)}. \quad (10)$$

In the following we summarize the relationships linking the κ -sums defined through Eqs. (8), (9) and (10)

$$q(u_1) \oplus q(u_2) = q(u_1 \oplus u_2), \quad (11)$$

$$\mathcal{E}(u_1) \oplus \mathcal{E}(u_2) = \mathcal{E}(u_1 \oplus u_2), \quad (12)$$

$$u(q_1) \oplus u(q_2) = u(q_1 \oplus q_2), \quad (13)$$

$$\mathcal{E}(q_1) \oplus \mathcal{E}(q_2) = \mathcal{E}(q_1 \oplus q_2), \quad (14)$$

$$q(\mathcal{E}_1) \oplus q(\mathcal{E}_2) = q(\mathcal{E}_1 \oplus \mathcal{E}_2), \quad (15)$$

$$u(\mathcal{E}_1) \oplus u(\mathcal{E}_2) = u(\mathcal{E}_1 \oplus \mathcal{E}_2). \quad (16)$$

Note that in the lhs and rhs of any of the above equations appear two different composition laws. For instance in Eq.(11) in the lhs appears the composition law of the dimensionless momenta (9) while in the rhs appears the composition law of the dimensionless velocities (8). We emphasize that the three κ -sums given by Eqs. (8), (9) and (10), emerge only when we change the frame of observation of the particle from \mathcal{S} to \mathcal{S}' .

III. LORENTZ INVARIANT INTEGRATION

Within the special relativity a central role is played by the four dimension Lorentz invariant integral

$$I_{rel} = A \int d^4 p \theta(p_0) \delta(p^\mu p_\mu - m^2 c^2) F, \quad (17)$$

with A an arbitrary constant and F an arbitrary function depending on $p = |p|$. After recalling that $p^\mu = (E/c, \mathbf{p})$ and $E = \sqrt{m^2 c^4 + p^2 c^2}$, one can reduce the above integral as follows

$$I_{rel} = \int \frac{d^3 q}{\sqrt{1 + \kappa^2 q^2}} F = \int_0^\infty \frac{dq}{\sqrt{1 + \kappa^2 q^2}} 4\pi q^2 F, \quad (18)$$

where the constant A is fixed properly and the dimensionless momentum is introduced. We remark that in (17) the integration element $d^4 p$ is a scalar because the Jacobian of the Lorentz transformation is equal to unity, so that I_{rel} transforms as F . Then in (18) the integration element $d^3 q / \sqrt{1 + \kappa^2 q^2}$ is a scalar. After introducing the κ -differential through

$$d_{\{\kappa\}} q = \frac{dq}{\sqrt{1 + \kappa^2 q^2}}. \quad (19)$$

the integral I_{rel} assumes the form

$$I_{rel} = \int_0^\infty d_{\{\kappa\}}q \, 4\pi q^2 F \quad . \quad (20)$$

One immediately observes that the Lorentz invariant integral can be obtained by deforming the ordinary (classical) integral

$$I_{cl} = \int_0^\infty dq \, 4\pi q^2 F \quad . \quad (21)$$

The deformation is obtained by performing the substitution $dq \rightarrow d_{\{\kappa\}}q$.

It is important to note that the κ -differential (19) can be obtained directly from the additivity law of the dimensionless relativistic momenta (9) according to

$$d_{\{\kappa\}}q = (q + dq) \stackrel{\kappa}{\ominus} q \quad . \quad (22)$$

IV. PHYSICAL MEANING OF THE κ -DIFFERENTIAL

We consider now two identical relativistic particles with rest mass m and velocities \mathbf{v}_1 and \mathbf{v}_2 respectively, in the frame \mathcal{S} . The modulus of the relative velocity $V = V(\mathbf{v}_1, \mathbf{v}_2)$ of the two particles, given in ref. [13] (page 20), can be written also in the form

$$V(\mathbf{v}_1, \mathbf{v}_2) = \sqrt{(\mathbf{v}_1 \ominus \mathbf{v}_2)^2 - \frac{1}{c^2} \left(\frac{\mathbf{v}_1 \times \mathbf{v}_2}{1 - \mathbf{v}_1 \mathbf{v}_2 / c^2} \right)^2} \quad , \quad (23)$$

with

$$\mathbf{v}_1 \ominus \mathbf{v}_2 = \frac{\mathbf{v}_1 - \mathbf{v}_2}{1 - \mathbf{v}_1 \mathbf{v}_2 / c^2} \quad . \quad (24)$$

We perform now the calculation of the modulus of relative momentum of the two particles according to $P(V) = mV/\sqrt{1 - V^2/c^2}$. Clearly, it results that $P(V) = P(\mathbf{v}_1, \mathbf{v}_2)$ and then $P(V) = P(\mathbf{p}_1, \mathbf{p}_1)$. At this point we introduce the dimensionless variables $\mathbf{q} = \mathbf{p}/mv_*$ and $Q = P/mv_*$. After some straightforward calculation, one arrives to the following expression for the modulus of the dimensionless relative momentum (the modulus of dimensionless momentum of the particle 1 (2) in the rest frame of other particle)

$$Q(\mathbf{q}_1, \mathbf{q}_2) = \sqrt{\frac{\left(\mathbf{q}_1 \stackrel{\kappa}{\ominus} \mathbf{q}_2 \right)^2 - \kappa^2 (\mathbf{q}_1 \times \mathbf{q}_2)^2}{1 + \kappa^4 (\mathbf{q}_1 \times \mathbf{q}_2)^2}} \quad , \quad (25)$$

with

$$\mathbf{q}_1 \stackrel{\kappa}{\ominus} \mathbf{q}_2 = \mathbf{q}_1 \sqrt{1 + \kappa^2 \mathbf{q}_2^2} - \mathbf{q}_2 \sqrt{1 + \kappa^2 \mathbf{q}_1^2} \quad . \quad (26)$$

The expression of the relative momentum given by Eq. (25) has been obtained in [12] (there the formula contains a typing error).

In the following, in order to explain the physical meaning of the κ -differential and the κ -derivative in special relativity, we consider that the two particles move along the same line so that $\mathbf{q}_1 \times \mathbf{q}_2 = 0$.

We suppose now that the two particles have momenta $\mathbf{q}_1 = \mathbf{q} + d\mathbf{q}$ and $\mathbf{q}_2 = \mathbf{q}$ respectively in the frame \mathcal{S} . To calculate the modulus of the momentum $Q(\mathbf{q} + d\mathbf{q}, \mathbf{q})$ of one particle in the rest frame of the other particle, after posing $q = |\mathbf{q}|$, one obtains:

$$Q(\mathbf{q} + d\mathbf{q}, \mathbf{q}) = d_{\{\kappa\}}q \quad . \quad (27)$$

The physical meaning of the κ -differential $d_{\{\kappa\}}q$ immediately follows. The infinitesimal difference dq of the momenta of the particle 1 and 2 in the frame \mathcal{S} becomes $d_{\{\kappa\}}q$ if this difference is observed in the rest frame of one of the two particles. The meaning of the κ -derivative

$$\frac{d}{d_{\{\kappa\}}q} = \sqrt{1 + \kappa^2 q^2} \frac{d}{dq} \quad , \quad (28)$$

readily follows. If d/dq represents the derivative in the frame \mathcal{S} , the deformed derivative $d/d_{\{\kappa\}}q$ represents an ordinary derivative in the rest frame of one of the two particles.

V. THE κ -EXPONENTIAL FUNCTION

We recall that the length of any four-vector is Lorentz invariant. In particular for the four-momentum we have the dispersion relation $p^\mu p_\mu = m^2 c^2$ which can be arranged as $(E/c - p)(E/c + p) = m^2 c^2$ being $p = |\mathbf{p}|$. This latter relationship after expressing the energy in terms of the moment according to $E = \sqrt{m^2 c^4 + p^2 c^2}$ becomes

$$\left(\sqrt{1 + \left(\frac{p}{mc} \right)^2} - \frac{p}{mc} \right) \left(\sqrt{1 + \left(\frac{p}{mc} \right)^2} + \frac{p}{mc} \right) = 1 \quad . \quad (29)$$

The latter equation, after introducing the dimensionless momentum q , can be also written as

$$\left(\sqrt{1 + \kappa^2 q^2} - \kappa q \right)^{1/\kappa} \left(\sqrt{1 + \kappa^2 q^2} + \kappa q \right)^{1/\kappa} = 1 \quad . \quad (30)$$

We remark that Eq. (30) follows directly from the relativistic dispersion relation. Interestingly, in the classical limit $\kappa \rightarrow 0$, Eq. (30) reduces to $\exp(-q) \exp(q) = 1$ while the dispersion relation becomes the classical one $W = p^2/2m$ being W the kinetic energy. In this way we obtain a direct link between the dispersion relation of free classical particles and the ordinary exponential function. In the light of the above result, we reconsider Eq. (30) written in the form

$$\exp_{\{\kappa\}}(-q) \exp_{\{\kappa\}}(q) = 1 \quad , \quad (31)$$

with

$$\exp_{\{\kappa\}}(q) = \left(\sqrt{1 + \kappa^2 q^2} + \kappa q \right)^{1/\kappa} \quad . \quad (32)$$

We feel that the function $\exp_{\{\kappa\}}(q)$, reproducing the ordinary exponential function $\exp(q) = \exp_{\{0\}}(q)$ in the classical limit $\kappa \rightarrow 0$, represents a one parameter relativistic generalization of the ordinary exponential.

Taking into account that the ordinary exponential results to be eigenfunction of the ordinary derivative, namely $(d/dq)\exp(q) = \exp(q)$, the question to determine the eigenfunction of the κ -derivative, naturally arises. After some simple calculation one obtains

$$\frac{d}{d_{\{\kappa\}}q} \exp_{\{\kappa\}}(q) = \exp_{\{\kappa\}}(q) \quad , \quad (33)$$

so that the κ -exponential emerges as eigenfunction of the κ -derivative $d/d_{\{\kappa\}}q$.

The κ -exponential has the following important property

$$\exp_{\{\kappa\}}(q_1) \exp_{\{\kappa\}}(q_2) = \exp_{\{\kappa\}}(q_1 \oplus_{\kappa} q_2) \quad , \quad (34)$$

being $q_1 \oplus_{\kappa} q_2$ the additivity law of the dimensionless relativistic momenta.

We note that Eq. (33) admits as unique solution the κ -exponential and then can be viewed as a differential equation defining it. Analogously Eq. (34) is a functional equation univocally defining the κ -exponential function. In conclusion we have discussed three different ways to introduce the κ -exponential. The first way is the more physical one employing the relativistic dispersion relation. The second and third ways are more mathematical and employ the differential equation (33) and the functional equation (34) respectively.

The mathematical properties of the κ -exponential can be found in refs. [10, 11, 12]. One important property of this function is its power law asymptotic behaviour when its argument tends to $\pm\infty$.

VI. THE RELATIVISTIC ENTROPY

The Boltzmann-Gibbs-Shannon entropy of a classical particle system is defined as the mean value of the minus logarithm of the distribution function, namely $S(f) = -\langle \ln(f) \rangle = -\int d^3p \, f \ln(f)$. In order to define the entropy of a relativistic particle system, we employ the same definition with the only difference that the ordinary logarithm now is replaced by the κ -logarithm, defined as the inverse function of the κ -exponential, namely

$$\ln_{\{\kappa\}}(f) = \frac{f^{\kappa} - f^{-\kappa}}{2\kappa} \quad . \quad (35)$$

The κ -logarithm is a one parameter continuous deformation of the ordinary logarithm which is recovered in the limit $\kappa \rightarrow 0$. The reason of the modification of the classical entropy is due to the fact that the κ -exponential and then the κ -logarithm naturally emerge within the special relativity at the place of the ordinary exponential and logarithm appearing in classical statistical mechanics.

The new entropy is given by

$$S(f) = -\langle \ln_{\{\kappa\}}(f) \rangle = -\int d^3p \, f \ln_{\{\kappa\}}(f) \quad , \quad (36)$$

and after maximization under the proper relativistic constraints, yields the distribution

$$f = \alpha \exp_{\{\kappa\}}\left(-\frac{(p^\nu + eA^\nu/c) U_\nu - mc^2 - \mu}{\lambda k_B T}\right) \quad , \quad (37)$$

with $\alpha = [(1 - \kappa)/(1 + \kappa)]^{1/2\kappa}$ and $\lambda = \sqrt{1 - \kappa^2}$. Being both these parameters real, $|\kappa| < 1$ follows. The above distribution results to be quite different with respect to the relativistic Maxwell-Boltzmann distribution where the κ -exponential is replaced by the ordinary exponential. The distribution (37), in the global rest frame where the hydrodynamic four vector velocity is $U^\nu = (c, 0, 0, 0)$ and in absence of external forces ($A^\nu = 0$), simplifies to

$$f = \alpha \exp_{\{\kappa\}}\left(-\frac{W - \mu}{\lambda k_B T}\right) \quad , \quad (38)$$

being W the relativistic kinetic energy. The latter distribution presents, for $W \rightarrow \infty$, power law tails, namely $f \propto W^{-1/\kappa}$, in contrast to the Maxwell-Gibbs distribution which decays exponentially. Moreover the distribution (38) fits very well the experimental data of the cosmic ray spectrum [11].

It is already known in the literature that the entropy (36) has the following properties: i) it is non-negative and achieves its maximum value at equiprobability, $f(p) = 1/\Omega$ for $\forall p$; and this value is $S = \ln_{\{\kappa\}}\Omega$; ii) it is concave so that the system is stable in thermodynamic equilibrium; iii) it satisfies the Lesche stability condition and then it is physically meaningful; iv) it generates a selfconsistent and coherent statistical mechanics. Furthermore it results that: v) this statistical mechanics can be obtained as stationary case of a kinetics governed by a generalized relativistic Boltzmann equation; vi) this generalized Boltzmann kinetics obeys the H-theorem which represents the second law of thermodynamics.

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